CLASSIFYING PROJECTIVE – LIKE HIERARCHIES

by Alexander S. Kechris[1)]

Dedicated to the memory of Christos D. Papakyriakopoulos

The starting point for the investigations reported in this paper is the following surprising result of Steel [St]:

Assuming AD + DC let Γ be a collection of pointsets containing all open sets and closed under continuous preimages and countable intersections and unions. Then either Γ or $\check{\Gamma}$ (= the dual of Γ) has the reduction property.

This result raises naturally the question if a similar phenomenon occurs with the stronger prewellordering property, perhaps under stronger closure assumptions on Γ. As a first step the following conjecture was proposed by Steel [St], the author and probably others:

From AD + DC (and maybe other hypotheses) if Γ is a collection of pointsets containing all the open sets and closed under continuous preimages, countable intersections and unions and universal and existential quantification over reals then either Γ or $\check{\Gamma}$ has the prewellordering property.

This conjecture was indeed proved in Kechris–Solovay [K–S] (for an exposition see [V 1]) assuming AD + DC + V = L[$\mathbb{R}$]. The proof was based on the following two facts of which the second needs V = L[$\mathbb{R}$] and explains clearly the role of this assumption.

1) (AD + DC) Let Γ be a pointclass which contains all open sets and is closed under continuous preimages, countable intersections and unions and existential and universal quantification over reals but not negation. Let $\check{\Gamma}$ be its dual class and put

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$\Delta = \Gamma \cap \breve{\Gamma}$. Then the following are equivalent:

i) Γ or $\breve{\Gamma}$ has the prewellordering property

ii) Δ is not closed under wellordered unions (i.e. there is $\{A_\xi\}_{\xi<\lambda} \subseteq \Delta$ such that $\bigcup_{\xi<\lambda} A_\xi \notin \Delta$).

This criterion was also independently proved by Steel.

2) If $V = L[\mathbb{R}]$, then every pointclass Λ which contains the open sets and is closed under continuous preimages, countable intersections and unions, quantification of both kinds over the reals and also negation and wellordered unions contains all pointsets.

In view of the requirement that the class Γ be closed under both existential and universal quantification over reals the preceding result does not cover the all important examples of the classical projective pointclasses and more generally the projective–like pointclasses i.e. pointclasses which have all the closure properties stated before except that they are closed under either existential or universal quantification over reals but not both. A natural extension of the previous conjecture should assert that, say assuming $AD + DC + V = L[\mathbb{R}]$, every projective–like pointclass Γ or its dual $\breve{\Gamma}$ has the prewellordering property. Moreover in view of the asymmetry between Γ and $\breve{\Gamma}$ one might also hope to determine in some sense the side on which the prewellordering property settles, as it is done by the work of Martin [Ma1] and Moschovakis [A–M] for the classical projective pointclasses. This is the main problem we address ourselves to in this paper.

Our basic approach is to utilize Martin's [Ma 3] basic result on the wellordering of Wadge degrees, which imposes a nice hierarchy on all the sets of reals (from AD + +DC). We have therefore started by using induction on the Wadge ordinal (of a complete set) of Γ, a projective–like pointclass for which we wish to establish the required properties. (This is in contrast with the proofs in Steel [St] and Kechris – Solovay [K–S] which operate directly on the pointclass Γ). As a result we are led naturally to consider the projective–like hierarchy in which Γ is embedded – the precise definition is given in Section 1. An analysis of these hierarchies leads to a classification of them into 4 basic Types I–IV. The classical projective hierarchy turns out to be of Type I. The propagation pattern of the reduction property in each Type is then studied and it is shown (from AD + DC + V = $L[\mathcal{R}]$) in Sections 3,4 that Types I, III exhibit the same pattern as the classical projective hierarchy, IV exhibits the opposite (dual) pattern and for Type II the problem is left open although we conjecture that it is like that of Type IV. Finally in Section 4 the pattern of the prewellordering property in each type of hierarchy is studied and it is shown that except for two possible exceptions (which we of course conjecture that do not exist) i.e. the first pointclass in a hierarchy of Type III and the first pointclass in a hierarchy of Type II that exhibits a reduction pattern dual to that of the classical projective hierarchy, the prewellordering property holds exactly where the reduction property does. Thus modulo these two kinds of exceptions we verify the conjecture that every projective–like pointclass or its dual satisfies the prewellordering property.

From the point of view of understanding the role of this work in the context of the definability theory of the continuum, this paper is a contribution to the study of a large class of definable sets of reals, namely those belonging in $L[\mathcal{R}]$, under the hypothesis of definable determinancy (in addition to the standard axioms of ZFC of course). It is our feeling that such studies of the "global" structure of extensive classes of definable sets are bound to increase our understanding of this powerful hypothesis. For our purposes here this can be understood as the hypothesis that every set of reals in $L[\mathcal{R}]$ is determined. That is the same thing as saying that AD (the Full Axiom of Determinacy) holds in $L[\mathcal{R}]$. Since the results we are interested in are absolute between $L[\mathcal{R}]$ and the real world we think of ourselves as living completely in $L[\mathcal{R}]$. For that reason we work throughout in this paper in ZF + AD + DC without further explicit mentioning. (Of course DC is the only form of AC that survives the tradition from the real world to $L[\mathcal{R}]$). When we additionally need the assumption $V = L[\mathcal{R}]$ we shall state it explicitly.

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1. Projective–like pointclasses and hierarchies.

Let $\omega = \{0,1,2,\ldots\}$ and $\mathcal{R} = \omega^{\omega}$ = the set of reals. We shall use letters i, j, k,... for elements of ω and α, β, γ ,... for reals. A space is a product of the form

$$\mathcal{X} = X_1 \times \ldots \times X_k ,$$

where each X_i is ω or $\mathcal{R}$. A pointset is a subset of a product space. We usually write interchangeably for $R \subseteq \mathcal{X}$

$$R(x) \Leftrightarrow x \in R.$$

A pointclass is a collection of pointsets usually in all product spaces.

1.1. Definition. A pointclass Γ is called *projective–like* if it satisfies the following conditions

i) It contains all the open sets (in all spaces) and is closed under countable intersections and unions and continuous substitutions (i.e. if $f: \mathcal{X} \to \mathcal{Y}$ is continuous and $A \subseteq \mathcal{Y}$ is in Γ, so is $f^{-1}[A]$).

ii) Γ is closed under $\forall^{\mathcal{R}}$ *or* $\exists^{\mathcal{R}}$ *but not both* (To say that Γ is closed under, say, $\forall^{\mathcal{R}}$ means that if $A \subseteq \mathcal{X} \times \mathcal{R}$ is in Γ so is $B = \{x: \forall \alpha\, A(x, \alpha)\}$).

iii) Γ is $\mathcal{R}$-parametrized i.e. for each space $\mathcal{X}$ there exists, $\mathfrak{X} \subseteq \mathcal{R} \times \mathcal{X}$ also in Γ such that if $\mathfrak{X}_\alpha = \{x: \mathfrak{X}(\alpha, x)\}$ then $\{\mathfrak{X}_\alpha : \alpha \in \mathcal{R}\} = \{A \subseteq \mathcal{X} : A \in \Gamma\}$. Such sets $\mathfrak{X}$ are called *universal.*

In view of Wadge's Lemma, $\mathcal{R}$ –parametrization is equivalent to non–closure of Γ under negation (see e.g. [V1]).

Of course the typical examples of projective–like pointclasses are the pointclasses $\underset{\sim}{\Sigma}^1_n$, $\underset{\sim}{\Pi}^1_n$, for $n \geqslant 1$.

In view of Wadge's Lemma again if Γ is a projective–like pointclass which is closed under $\exists^{\mathcal{R}}$ but not $\forall^{\mathcal{R}}$ then $\forall^{\mathcal{R}}\Gamma = \{\forall \alpha\, A(x, \alpha) : A \in \Gamma\}$ contains both Γ and $\breve{\Gamma} = \{\mathcal{X} - A : A \subseteq \mathcal{X}, A \in \Gamma\}$ and is also a projective–like pointclass. Similarly if Γ is closed under $\forall^{\mathcal{R}}$ but not under $\exists^{\mathcal{R}}$ then

$\exists^{\mathcal{R}}\Gamma = \{\exists \alpha \;\; A(x,\alpha) : A \in \Gamma\}$ is a projective–like pointclass containing both Γ and $\breve{\Gamma}$. Note that if $\Game$ is the game quantifier $\Game \alpha A(x, \alpha) \Rightarrow \exists \alpha(0) \;\; \forall \alpha(1) \ldots A(x, \alpha)$ and $\Game\Gamma = \{\Game \alpha A(x, \alpha): \;\; A \in \Gamma\}$ then for any projective–like pointclass Γ:

$$\Game\Gamma = \begin{cases} \forall^{\mathcal{R}}\Gamma, & \text{if } \Gamma \text{ is closed under } \exists^{\mathcal{R}}, \\ \\ \exists^{\mathcal{R}}\Gamma & \text{if } \Gamma \text{ is closed under } \forall^{\mathcal{R}}. \end{cases}$$

1.2. Definition. *A projective–like hierarchy* is a sequence $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ of pointclasses such that

i) Each Γ_i is a projective–like pointclass and Γ_1 is closed under $\forall^{\mathcal{R}}$.

ii) $\Gamma_{i+1} = \Game\Gamma_i$.

iii) $\{\Gamma_i\}$ is maximal i.e. for no projective–like Γ_0 closed under $\exists^{\mathcal{R}}$ we have $\Game\Gamma_0 = \forall^{\mathcal{R}}\Gamma_0 = \Gamma_1$

We will say that a pointclass Γ *belongs* to the hierarchy $\{\Gamma_i\}$ if for some i, $\Gamma = \Gamma_i$ or $\Gamma = \breve{\Gamma}_i$.

The standard projective-like hierarchy is of course the classical one: $\underset{\sim}{\Pi}{}^1_1$, $\underset{\sim}{\Sigma}{}^1_2$, $\underset{\sim}{\Pi}{}^1_3$, $\underset{\sim}{\Sigma}{}^1_4, \ldots$.

For further use let us also introduce the following notions and terminology:

1) For each $A \subseteq \mathcal{R}$ there is a smallest projective–like pointclass closed under $\forall^{\mathcal{R}}$ which contains A – see [Mo. 1] . It will be denoted by $\underset{\sim}{\Pi}{}^1_1(A)$.

2) A pointclass Γ which contains all the open sets and is closed under complements, continuous substitutions, finite intersections and unions and $\exists^\omega, \forall^\omega, \exists^{\mathcal{R}}, \forall^{\mathcal{R}}$ will be called strongly *closed.* (Closure of Γ under $\exists^\omega$ means of course the obvious thing: if $A \subset X \times \omega$ is in Γ then so is $\exists n\, A(x,n)$ and similarly for $\forall^\omega$). The typical example of strongly closed Γ is of course the pointclass of all projective sets. Note that a strongly closed pointclass does not have to be closed under countable intersections or unions.

2. Types of projective–like hierarchies.

We shall introduce here a classification of all projective hierarchies into four basic types. For that we shall need some facts and notation from the theory of Wadge degrees for which the reader is referred to [V1].

For $A, B \subseteq \mathcal{R}$ let

$$A \leqslant_w B \Leftrightarrow \text{There is a continuous } f \text{ such that } A = f^{-1}[B].$$

By a theorem of Martin [Ma 3] the relation

$$A \leqslant_w B \text{ or } A \leqslant_w \neg B (= \mathcal{R} - B)$$

is a prewellordering on power $(\mathcal{R})$, so for each $A \subseteq \mathcal{R}$ let

$$|A|_w \equiv \text{Wadge ordinal of } A$$

$$\stackrel{\text{def}}{\equiv} \text{the ordinal of A in this prewellordering.}$$

Call A *self–dual* if $A \leqslant_w \neg A$. The following result of Steel and independently Van Wesep (see [V 1]) will be instrumental in the following

2.1. Theorem (Steel, Van Wesep [V1]).

i) If $A \subseteq \mathcal{R}$ is self–dual, then either $|A|_w$ is a successor ordinal or a limit of cofinality ω.

ii) If $|A|_w$ is a limit ordinal of cofinality ω, then A is self–dual.

iii) If A is self–dual and $|B|_w = |A|_w + 1$, then B is not slef–dual, while if A is not self–dual and $|B|_w = |A|_w + 1$ then B is self–dual.

Let us now introduce the four types of projective–like hierarchies:

Type I: $\{\Gamma_i\}$ is such that $\Gamma_1 = \underset{\sim}{\Pi}_1^1(A)$ for some A such that $|A|_w$ has cofinality ω and $\Lambda = \{B : |B|_w < |A|_w\}$ is strongly closed.

The classical projective hierarchy is included in this type as the degenerate case $|A|_w = 0$. Another interesting example results bv taking A to be such that

$\{B : |B|_w < |A|_w\}$ = all projective sets.

Type II: $\{\Gamma_i\}$ is such that $\Gamma_1 = \underset{\sim}{\Pi}{}^1_1(A)$, where $|A|_w$ has cofinality $>\omega$, and $\neg A \in \underset{\sim}{\Pi}{}^1_1(A)$, and $\Lambda = \{B : |B|_w < |A|_w\}$ is strongly closed.

For the simplest example of this type let A be such that $\{B : |B|_w < |A|_w\} = \cup_\eta \Gamma_\eta$, where Γ_η for $1 \leqslant \eta \leqslant \omega_1$ is defined as follows

$$\Gamma_1 = \underset{\sim}{\Pi}{}^1_1$$

$$\Gamma_{\eta+1} = \Game\, \Gamma_\eta$$

$$\Gamma_\lambda = \text{all countable unions of pointsets in } \bigcup_{\eta<\lambda} \Gamma_\eta,$$

$$\text{if } \lambda \text{ is limit.}$$

Type III: $\{\Gamma_i\}$ is such that $\Delta_1 = \Gamma_1 \cap \breve{\Gamma}_1$ is strongly closed.

For an important example let Γ_1 = all pointsets Kleene semirecursive in ${}^3E$ and a real.

Type IV: $\{\Gamma_i\}$ is such that $\Gamma_1 = \forall^{\mathcal{R}}(\Gamma \mathring{\cup} \breve{\Gamma})$, where Γ is *inductive–like* i.e. it is a pointclass containing all open sets, is closed under continuous substitutions, countable intersections and unions and both $\exists^{\mathcal{R}}$ and $\forall^{\mathcal{R}}$ and is also $\mathcal{R}$–parametrized. A typical example (the smallest such actually) is the class Γ of all inductive on the structure of analysis pointsets, also called the *semi–hyperprojective* sets, see [Mo 2]. Here $\Gamma \mathring{\cup} \breve{\Gamma} = \{A \cup B : A \in \Gamma,\ B \in \dot{\Gamma}\}$.

The fact that this is a complete classification of projective–like hierarchies is our first result here.

2.2. Theorem: Every projective–like hierarchy is of exactly one of the types I – IV.

Proof. Clearly every projective–like pointclass Γ belongs to exactly one projective–like hierarchy $\{\Gamma_i\}$. Indeed, either for no projective–like Γ_0, $\Game\,\Gamma_0 = \Gamma$.

in which case, granting say that Γ is closed under $\forall^{\mathcal{R}}$, we must have $\Gamma_1 = \Gamma$, $\Gamma_2 = \partial\Gamma$, $\Gamma_3 = \partial\Gamma_2, \ldots$, or there is a projective–like Γ' such that $\partial\Gamma' = \Gamma$. Now calling a set $A \subseteq \mathcal{R}$ *complete* in a pointclass Σ if for every $B \subseteq \mathcal{R}$, $B \in \Sigma$ we have $B \leqslant_w A$, we observe that if A is complete for Γ' and B is complete for Γ then $|A|_w < |B|_w$ since $\Gamma' \cup \breve{\Gamma}' \subseteq \Gamma$. As a result we can assume by induction on the Wadge ordinal of a complete set in a projective–like pointclass $\Gamma \equiv^{\mathrm{def}}$ the *Wadge ordinal* of $\Gamma \equiv |\Gamma|_w$ that Γ' belongs to a projective–like hierarchy $\{\Gamma_i\}$. Then of course Γ belongs in the same hierarchy. Uniqueness now follows from the following fact: If Γ^*, Γ^{**} are projective–like pointclasses both closed under $\exists^{\mathcal{R}}$ (or both under $\forall^{\mathcal{R}}$) and $\partial\Gamma^* = \partial\Gamma^{**}$ then $\Gamma^* = \Gamma^{**}$. Indeed if this is not true and A^*, A^{**} are complete sets in Γ^*, Γ^{**} resp. then we must have that $|A^*|_w < |A^{**}|_w$ or $|A^{**}|_w < |A^{**}|_w$; say the first case occurs. Then $\breve{\Gamma}^* \subseteq \Delta^{**}$. So $\partial\Gamma^{**} = \partial\Gamma^* = \forall^{\mathcal{R}}\Gamma^* = (\overbrace{\exists^{\mathcal{R}}(\breve{\Gamma}^*)}) \subseteq (\overbrace{\exists^{\mathcal{R}}\Delta^{**}}) \subset \overbrace{\Gamma^{**}}$, a contradiction.

By this preliminary discussion it is sufficient to prove that every projective–like pointclass Γ belongs to a projective–like hierarchy of one of these four types. Since it is obvious that no such hierarchy can belong to more than one of these types this will complete the proof. We assume that for all Γ' projective–like pointclasses with $|\Gamma'|_w < |\Gamma|_w$ the result holds and we consider three cases. We assume w.l.o.g. that Γ is closed under $\forall^{\mathcal{R}}$.

Case 0. Γ belongs to the classical projective hierarchy. Then we are done.

Case A. Case 0 fails and $\Delta = \Gamma \cap \breve{\Gamma}$ is strongly closed. Then $\Gamma = \Gamma_1$ for a projective–like hierarchy of Type IV.

Case B. Cases O, A fail. Then there must be some $A \in \Delta$, so that $\Gamma = \underset{\sim}{\Pi}^1_1(A)$. Pick such an A of least Wadge ordinal.

Subcase B.1. $|A|_w$ is limit.

Then we have to distinguish between

B.1.1. cofinality $(|A|_w) = \omega$

and

B.1.2. cofinality ($|A|_w$) $> \omega$.

Under B.1.1., A is self–dual by 2.1 and if $|B|_w < |A|_w$ then $\underset{\sim}{\Pi}^1_1(B) \subseteq \{B' : |B'|_w < |A|_w\}$ by the minimality of $|A|_w$, so $\{|B'| : |B'.|_w < |A|_w\}$ is strongly closed, thus $\Gamma = \Gamma_1$ for a projective–like hierarchy of Type I.

Under B.1.2. A is not self–dual so either for some $|B|_w < |A|_w$ we have $\underset{\sim}{\Sigma}^1_1(B) = \{A' : A' \leqslant_w A\}$ so that if $\Gamma' = \underset{\sim}{\Sigma}^1_1(B)$, we have $\Gamma = \Game\Gamma'$ and we can apply our induction hypothesis since $|\Gamma'|_w < |\Gamma|_w$ or $\{B : |B|_w < |A|_w\}$ is strongly closed. But then either $\neg A \in \underset{\sim}{\Pi}^1_1(A)$ in which case $\Gamma = \Gamma_1$ for some hierarchy $\{\Gamma_i\}$ of type II or $\Gamma = \underset{\sim}{\Pi}^1_1(A) = \{A' : A' \leqslant_w A\}$ a contradiction since $A \in \Delta$. Thus we have completed the proof in Subcase B.1.

Subcase B.2. $|A|_w$ is successor.

Here again we have to distringuish between

B.2.1. A is self–dual and

B.2.2. A is not self-dual.

Under B.2.1. we have 2.1 that $A = (0 * B) \cup (1 * \neg B)$, where $|B|_w + 1 = |A|_w$ and B is not self–dual (Here $i * X = \{ i^\frown a : a \in X \}$). Consider now $\underset{\sim}{\Pi}^1_1(B)$. If $\neg B \in \underset{\sim}{\Pi}^1_1(B)$ then $A \in \underset{\sim}{\Pi}^1_1(B)$, so $\underset{\sim}{\Pi}^1_1(B) = \Gamma$ contradicting the minimality of $|A|_w$. So $\underset{\sim}{\Pi}^1_1(B) = \{B : B' \leqslant_w B\}$ and similarly $\underset{\sim}{\Pi}^1_1(\neg B) = \{B' : B' \leqslant_w \neg B)$. So $\Gamma_0 = \underset{\sim}{\Pi}^1_1(B)$ is inductive–like and $\Gamma = \forall^{\mathbb{R}}(\Gamma_0 \ \mathring{\cup} \ \breve{\Gamma}_0)$, thus $\Gamma = \Gamma_1$ for some $\{\Gamma_i\}$ of type IV. Under B.2.2., there is some self-dual B with $|B|_w + 1 = |A|_w$, by 2.1 again and it is not hard to see that $A \in \underset{\sim}{\Pi}^1_1(B)$, so $\Gamma = \underset{\sim}{\Pi}^1_1(B)$, contradicting the minimality of $|A|_w$.

The proof of the theorem is now complete.

3. Characters of projective–like hierarchies.

One of the most important characteristics of the structure theory of the classical projective hierarchy is the propagation behaviour of the reduction property. It holds for exactly the pointclasses $\underset{\sim}{\Pi}^1_1$, $\underset{\sim}{\Sigma}^1_2$, $\underset{\sim}{\Pi}^1_3$, $\underset{\sim}{\Sigma}^1_4$, ..., as it follows from a result of Martin [Ma 1] and Moschovakis [A-M].

In view of that phenomenon we are led to the following definition.

3.1. Definition Let $\{\Gamma_i\}$ be a projective – like hierarchy. We say that $\{\Gamma_i\}$ is of *character* Π if each Γ_i has the reduction property and that $\{\Gamma_i\}$ is of *character* Σ if each $\breve{\Gamma}_i$ has the reduction property.

Thus the classical projective hierarchy is of character Π.

3.2. Theorem. Let $\{\Gamma_i\}$ be a projective–like hierarchy. Then $\{\Gamma_i\}$ is of exactly one of the characters Π or Σ.

Proof. The proof that $\{\Gamma_i\}$ cannot be both of character Π and Σ is an obvious consequence of the wellknown fact that if Γ is a projective–like pointclass with the reduction property, then $\breve{\Gamma}$ does not have the reduction property. The proof that $\{\Gamma_i\}$ is character Π or Σ is an immediate consequence of the following two results. The first is:

3.3. Theorem (Steel [St]). Let Γ be a pointclass containing all the open sets which is closed under continuous substitutions, countable intersections and unions and is $\mathcal{R}$ – parametrized. Then either Γ or $\breve{\Gamma}$ has the reduction property

To state the second we need some more concepts and notation

A pointset $P(x,\alpha)$ is called T– *invariant* (on α) if

$$P(x,\alpha) \wedge \alpha \equiv_T \beta \Rightarrow P(x,\beta),$$

where $\alpha \leqslant_T \beta \Leftrightarrow \alpha$ is recursive in β, and $\alpha \equiv_T \beta \Leftrightarrow \alpha \leqslant_T \beta \wedge \beta \leqslant_T \alpha$ For $P(x,\alpha)$ T–invariant we have by a well–known result of Martin:

$$\Game\, \alpha P(x,\alpha) \Leftrightarrow \{\alpha : P(x,\alpha)\} \text{ contains a cone of Turing degrees}$$

$$(\text{i.e. } \exists \alpha \forall \beta \geqslant_T \alpha\ P(x,\beta))$$

For any pointclass Γ let

$$\Game^* \Gamma = \{\Game \alpha\, P(x,\alpha) : P \in \Gamma,\ P \text{ T–invariant}\}$$

Then it is easy to check that if Γ contains all the open sets and is closed under continuous substitutions, countable intersections and unions and either $\exists^{\mathcal{R}}$ or $\forall^{\mathcal{R}}$ then $\Game\Gamma = \Game^*\Gamma$. This follows immediately from the following two equivalences

$$\exists\alpha\, P(x,\alpha) \Leftrightarrow \Game\alpha\exists\beta \leqslant_T \alpha\, P(x,\alpha)$$

$$\forall\alpha\, P(x,\alpha) \Leftrightarrow \Game\alpha\forall\beta \leqslant_T \alpha\, P(x,\alpha)$$

We now have

3.4. Theorem (The 0^{th} Periodicity Theorem) Let Γ be a pointclass containing all the open sets and closed under continuous substitutions and countable intersections and unions. Then if Γ has the reduction property so does $\Game^*\Gamma$.

Proof. Let $P, Q \in \Game^*\Gamma$ say $P(x) \Leftrightarrow \Game\alpha\, A(x,\alpha)$, $Q(x) \Leftrightarrow \Game\alpha\, B(x,\alpha)$, where $A, B \in \Gamma$ are T–invariant. By a result of Burgess and Miller [B–G] let $A_1 \subseteq A$, $B_1 \subseteq B$ be two T–invariant pointsets in Γ reducing A,B. Put $P_1(x) \Leftrightarrow \Game\alpha\, A_1(x,\alpha)$, $Q_1(x) \Leftrightarrow \Game\alpha\, B_1(x,\alpha)$. Clearly $P_1, Q_1 \in \Game^*\Gamma$ and one can easily check that P_1, Q_1 reduce P,Q.

To complete the proof of Theorem 3.2 let $\{\Gamma_i\}$ be a projective–like hierarchy. By 3.3 either Γ_1 or $\breve{\Gamma}_1$ has the reduction property. If it is Γ_1 then by 3.4. each Γ_i has the reduction property while in the other case each $\breve{\Gamma}_i$ has the reduction property.

What is the relation between the type and the character of a projective–like hierarchy? Our conjecture is that every type has a uniquely associated character. We have been able to prove this however only for the three out of the four types.

3.5. Theorem.

i) Any projective–like hierarchy of Type I is of character Π.

ii) Any projective–like hierarchy of Type III is of character Π.

iii) Any projective–like hierarchy of Type IV is of character Σ.

Conjecture. Any projective–like hierarchy of Type II is of character Σ.

Some (but far from totally convincing) evidence for this conjecture is offered by the following partial results:

Let A be as in the definition of a projective–like hierarchy of Type II. Put

$$\lambda = \sup \{ \xi : \xi \text{ is the length of a prewellordering of } \mathcal{R} \text{ in } \Lambda\}.$$

Then the conjecture holds if *either* of the conditions below holds

i. $\{C : C \leqslant_w \neg A\}$ and $\{C : C \leqslant_w A\}$ are closed under finite unions and one of them has the separation property.

ii. λ is not weakly inaccessible.

We shall postpone the proofs of these results until the next section.

4. ON THE PREWELLORDERING PROPERTY.

We shall now examine the behaviour of projective–like classes relative to the prewellordering property. The following two results should be mentioned first.

4.1. Theorem (Classical). If Γ has the prewellordering property then Γ has the reduction property. (We are assuming here that Γ contains all the open sets and is closed under continuous substitutions and finite intersections and unions).

4.2. Theorem (The First Periodicity Theorem).

i) (Moschovakis [Mo 2]) If the pointclass Γ contains all the open sets and is closed under continuous substitutions, finite intersections and unions and $\forall^{\mathcal{R}}$ and has the prewellordering property, then $\exists^{\mathcal{R}}\Gamma$ has the prewellordering property.

ii) (Martin [Ma 1], Moschovakis [A–M]). If the pointclass Γ contains all the open sets and is closed under continuous substitutions, finite intersections and unions and $\exists^{\mathcal{R}}$ and has the prewellordering property, then $\forall^{\mathcal{R}}\Gamma$ has the prewellordering property.

In particular if Γ is a projective–like pointclass with the prewellordering property then $\Game\Gamma$ has the prewellordering property.

In view of these two theorems the possible prewellordering pattern of projective–like hierarchies is clear: If $\{\Gamma_i\}$ is of type Π then the prewellordering property can only hold for the Γ_i's and if it holds for some Γ_{i_0} then it holds for all Γ_i's with $i \geqslant i_0$, while if $\{\Gamma_i\}$ is of type Σ then the same conclusion with Γ_i replaced by $\breve{\Gamma}_i$ holds.

Conjecture. ($V = L[\mathcal{R}]$). For every projective–like pointclass Γ either Γ or $\breve{\Gamma}$ has the prewellordering property.

A consequence of this conjecture is of course that if $\{\Gamma_i\}$ is of type Π then all Γ_i have the prewellordering property and if $\{\Gamma_i\}$ is of type Σ all $\breve{\Gamma}_i$ have the prewellordering property. So if this conjecture is true one can determine given a projective–like pointclass Γ on which side (Γ or $\breve{\Gamma}$) the prewellordering property settles.

On the basis of our next result this conjecture is confirmed except for two exceptions.

4.3. Theorem. ($V = L[\mathcal{R}]$). Let Γ be a projective–like pointclass. Then either Γ or $\breve{\Gamma}$ has the prewellordering property except possibly for two exceptions:

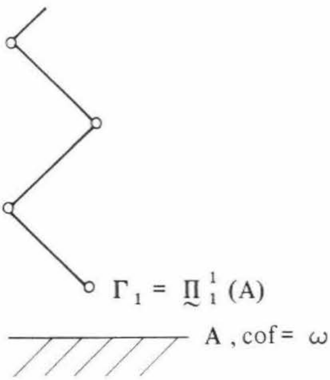
i. $\Gamma = \Gamma_1$ for a projective–like hierarchy of Type II and character Π.

ii. $\Gamma = \Gamma_1$ for a projective–like hierarchy of Type III.

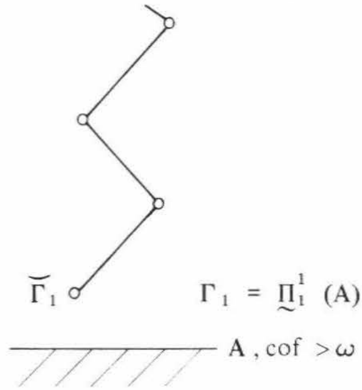
Corollary ($V = L[\mathcal{R}]$). If Γ is a projective–like hierarchy then either $\Game\Gamma$ or $\widetilde{\Game\Gamma} = \Game\breve{\Gamma}$ has the prewellordering property.

Before we proceed to the proofs of 4.3 and 3.5 (as well as the remarks following it) we will summarize their conclusions and the remaining open cases in the following picture.

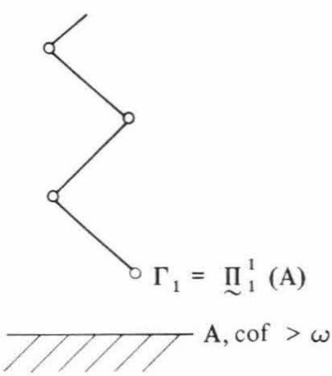
TYPE I – CHAR. Π



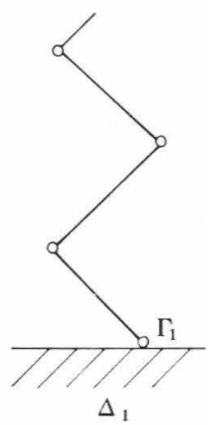
TYPE II – CHAR. Σ



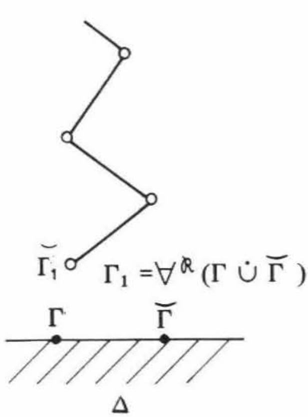
TYPE II – CHAR. Π (Exists ?) (PWO (Γ_1)?)



TYPE III – CHAR. Π (PWO (Γ_1)?)



TYPE IV – CHAR. Σ



The proofs of Theorems 4.3 and 3.5 will be split in a sequence of four lemmas from which the full results are immediate.

Lemma I. Every hierarchy $\{\Gamma_i\}$ of Type I is of character Π and every pointclass Γ_i has the prewellordering property.

Proof. Let A be as in the definition of a Type I hierarchy so that $\Gamma_1 = \underset{\sim}{\Pi}{}_1^1(A)$. We can of course assume that $|A|_w > 0$, since otherwise the result is well–known. Let $\Lambda = \{B : |B|_w < |A|_w\}$. Then since cofinality $(|A|_w) = \omega$, Λ is not closed under countable unions so let $\Gamma_0 = \{\bigcup_n A_n : A_n \in \Lambda\}$. Now it is easy to check that Γ_0 contains all the open sets, is closed under continuous substitutions, finite intersections and unions and $\exists^{\mathcal{R}}$ and has the prewellordering property so by 4.2, ii) $\Gamma_1 = \forall^{\mathcal{R}}\Gamma_0$ has the prewellordering property and we are done.

Lemma II. ($V = L[\mathcal{R}]$). Let $\{\Gamma_i\}$ be a hierarchy of Type II. Then either $\{\Gamma_i\}$ has character Σ and every $\breve{\Gamma}_i$ has the prewellordering property or $\{\Gamma_i\}$ has character Π and every Γ_i with $i \geqslant 2$ has the prewellordering property.

Proof. Let A again be as in the definition of a Type II hierarchy so that $\Gamma_1 = \underset{\sim}{\Pi}{}_1^1(A)$. Since $\Lambda = \{B : |B|_w < |A|_w\}$ is strongly closed we have by Kechris–Solovay [K–S] that for some $\{A_\xi\}_{\xi<\theta} \subseteq \Lambda$, $\bigcup_{\xi<\theta} A_\xi \notin \Lambda$.

Let θ_0 = least ordinal for which there is a sequence $\{A_\xi\}_{\xi<\theta} \subseteq \Lambda$ with $\bigcup_{\xi<\theta} A_\xi \notin \Lambda$. Then θ_0 is a regular cardinal $> \omega$ and we can find a sequence $\{A_\xi\}_{\xi<\theta_0} \subseteq \Lambda$ with $\bigcup_{\xi<\theta_0} A_\xi \notin \Lambda$ such that moreover $A_\xi \cap A_\eta = \phi$ if $\xi \neq \eta < \theta_0$ and $A_\xi \neq \phi$, $\forall \xi < \theta_0$. As a consequence $\theta_0 \leqslant \lambda$ = = sup $\{\xi : \xi$ is the length of a prewellordering in $\Lambda\}$. Because if $\theta < \theta_0$ then the prewellordering

$$a \leqslant \beta \Leftrightarrow \exists \xi \exists \eta (\xi < \eta < \theta \wedge a \in A_\xi \wedge \beta \in A_\eta)$$

is in Λ by the minimality of θ_0. Since $\lesssim$ has length θ we have $\theta < \lambda$, so $\theta_0 \leqslant \lambda$. Put (following an idea of Martin [Ma 2])

$$\Gamma^* = \{ \bigcup_{\xi < \theta_0} B_\xi : B_\xi \in \Lambda, \ \forall \xi < \theta_0 \}.$$

It is not hard to see that Γ^* has the prewellordering property. Indeed, if for $a \in \bigcup_{\xi<\theta_0} B_\xi$ we let $\sigma(a)$ = least $\xi < \theta_0$ such that $a \in B_\xi$, it is immediate (using the minimality of θ_0) that σ is a Γ^* – norm.

We claim now that

$$(*) \qquad \Gamma^* \subseteq \Gamma_2 = \exists^{\mathcal{R}} \Gamma_1.$$

Granting this claim we have that either $\Gamma^* = \{ B : B \leqslant_w A \}$ which is impossible since Γ^* is projective-like (closed under $\exists^{\mathcal{R}}$) while $\Gamma_1 = \forall^{\mathcal{R}} \Gamma^*$ or $\Gamma^* = \{ B : B \leqslant_w \neg A \}$, which implies that $\{ B : B \leqslant_w A \}$ is closed under continuous preimages and countable intersections, unions and $\forall^{\mathcal{R}}$, a contradiction since then $\Gamma_1 = \{ B : B \leqslant_w A \}$ so $\neg A \notin \Gamma_1$ or finally both A, $\neg$ A are in Γ^* so since Γ^* is projective–like and closed under $\exists^{\mathcal{R}}$, $\breve{\Gamma}_1 \subseteq \Gamma^*$. Therefore, either $\breve{\Gamma}_1 = \Gamma^*$ so that $\{\Gamma_i\}$ is of character Σ and every $\breve{\Gamma}_i$ has the prewellordering property or by Wadge's Lemma $\Gamma_1 \subseteq \Gamma^*$ so $\Gamma_2 \subseteq \Gamma^*$ and thus by (*) $\Gamma_2 = \Gamma^*$ and $\{\Gamma_i\}$ is of character Π and all Γ_i with $i \geqslant 2$ have the prewellordering property.

To prove (*) we use the following strong version of Moschovakis' Theorem in [Mo 1], which is also due to Moschovakis (its proof is similar to that in [Mo 1]).

Theorem (Moschovakis). If $<$ is a wellfounded relation on $\mathcal{R}$ of length ξ and if $\{A_\theta\}_{\theta < \xi}$ is a sequence of sets in $\underset{\sim}{\Sigma}^1_1(<)$ then $\bigcup_{\theta < \xi} A_\theta \in \underset{\sim}{\Sigma}^1_1(<)$.

In view of this result it is enough to produce a wellfounded relation in Γ_2 of length $\geqslant \lambda$. But this is easy:

Let

$\langle \epsilon , \alpha \rangle < \langle \epsilon' , \beta \rangle \Leftrightarrow \epsilon = \epsilon' \wedge \epsilon$ codes a continuous function $f_\epsilon : \mathcal{R} \to \mathcal{R}$
and $\{ (\alpha' , \beta') : \langle \alpha' , \beta' \rangle \in f_\epsilon^{-1} [A] \}$ is wellfounded and $\langle \alpha, \beta \rangle \in f_\epsilon^{-1} [A]$.

Clearly $< \in \Gamma_2$ and length $(<) \geqslant \lambda$, so we are done.

Remark. From the arguments just given it is clear that the second possibility in Lemma II would be eliminated (according to our conjecture) if we could find a wellfounded relation of length $\geqslant \theta_0$ (we retain the notation of the preceding proof) in $\breve{\Gamma}_1$. If $\theta_0 < \lambda$ this is clear so we can assume that $\theta_0 = \lambda$. Thus λ is a regular cardinal. But by Theorem 4 in Moschovakis [Mo 1] λ is a limit cardinal. So λ is weakly inaccessible. This establisches the second assertion at the end of Section 3.

To establish the first assertion there, let us assume without loss of generality that $\Pi = \{ B : B \leqslant_{\overline{w}} \neg A \}$ has the separation property. Let then P,Q be two disjoint sets in $\exists^{\mathcal{R}} \Pi$. Write

$$x \in P \Leftrightarrow \exists \alpha A (x,\alpha)$$

$$x \in Q \Leftrightarrow \exists \beta B (x, \beta)$$

where $P,Q \in \Pi$ are also disjoint. Put

$$A' (x, \alpha, \beta) \Leftrightarrow A (x,\alpha)$$

$$B' (x, \alpha, \beta) \Leftrightarrow B (x,\beta)$$

Then $A', B' \in \Pi$ and they are disjoint so let $C \in \Pi \cap \breve{\Pi} = \{B : |B|_w < |A|_w = \Lambda$ separate A', B' . Then if

$$x \in S \Leftrightarrow \exists \alpha \forall \beta\ C(x, \alpha, \beta),$$

$S \in \Lambda$ by the strong closure of Λ and S separates P,Q. Thus $\exists^{\mathcal{R}} \Pi$ cannot contain A (otherwise $A \in \Lambda$) so by Wadge's Lemma

$$\exists^{\mathcal{R}} \Pi = \{B : B \leqslant_w \neg A\}$$

thus $\breve{\Pi} = \{B : B \leqslant_w A\}$ is closed under $\forall^{\mathcal{R}}$. Now by a result of Van Wesep [V 2] Π satisfies the second separation property i.e. for any $A_0, B_0 \in \Pi$ there are $A_0', B_0' \in \breve{\Pi}$ disjoint such that $A_0 - B_0 \subseteq A_0'$, $B_0 - A_0 \subseteq B_0'$. Let $W^0, W^1 \subseteq \mathcal{R}^2$ be a universal pair of pointsets in Π i.e. $W^0, W^1 \in \Pi$ and for every $X, Y \subseteq \mathcal{R}$ in Π, there is $\epsilon \in \mathcal{R}$ such that

$$X = W^0_\epsilon \left(\stackrel{\text{def}}{=} \{\alpha : (\epsilon, \alpha) \in W^0\} \right)$$

$$Y = W^1_\epsilon.$$

By second separation let $\bar{W}^0, \bar{W}^1 \in \breve{\Pi}$ be such that $W^0 - W^1 \subseteq \bar{W}^0$, $W^1 - W^0 \subseteq \bar{W}^1$ and $\bar{W}^0 \cap \bar{W}^1 = \phi$. Using $\bar{W}^0, \bar{W}^1$ we can now define a coding system for Λ sets as follows:

$$\epsilon \in \text{Code} \Leftrightarrow \bar{W}^0_\epsilon \cup \bar{W}^1_\epsilon = \mathcal{R}$$

and for $\epsilon \in$ Code we let

$$\Lambda_\epsilon = \bar{W}^0_\epsilon .$$

Clearly $\Lambda = \{\Lambda_\epsilon : \epsilon \in \text{Code}\}$ and Code $\in \breve{\Pi}$.

Note now that $\sup \{ |B|_w : B \in \Lambda\} = \lambda$. This is because if $\leqslant$ is a prewellordering in Λ we can find a projective–like pointclass $\Sigma \subseteq \Lambda$ closed under $\exists^{\mathcal{R}}$, having the prewellordering property and containing $\leqslant$ (this follows easily from Lemma I). Then by a result of Martin and (independently) Steel (see [V 1])

$\sup\{|X|_w : X \in \Sigma \cap \breve{\Sigma}\} >$ length $(\leqslant)$ and we are done.

To complete the proof let $\{B_\xi\}_{\xi < \theta_0} \subseteq \Lambda$. We have to show that $\bigcup_{\xi < \theta_0} B_\xi \subseteq \breve{\Gamma}_1$.

For that consider the following game:

$$\begin{array}{lll} \text{I} & \text{II} & \text{I plays } \epsilon, \text{II plays } a \\ \epsilon & a & \text{and II wins iff} \end{array}$$

$$\epsilon \in \text{Code} \Rightarrow a \in \text{Code} \wedge \exists\, \xi\,(\ |\Lambda_\epsilon|_w < \xi < \theta_0 \ \text{ and } \ \Lambda_a = \bigcup_{\eta < \xi} B_\eta).$$

It is enough to prove that II has a winning strategy, say σ. Because then

$$x \in \bigcup_{\xi < \theta_0} B_\xi \Leftrightarrow \quad \exists\, \epsilon\,(\epsilon \in \text{Code} \wedge x \in \Lambda_{\sigma[\epsilon]}),$$

so $\bigcup_{\xi < \theta_0} B_\xi \in \breve{\Gamma}_1$ and we are done. Thus assume I has a winning strategy τ, towards a contradiction. Then for any a, $\tau[a] \in$ Code. Thus $R(a, x) \Leftrightarrow x \in \Lambda_{\tau[a]}$ is in Λ. So let $a_0 \in$ Code be such that for some $\theta_0 > \xi_0 > |R|_w$, $\Lambda_{a_0} = \bigcup_{\eta < \xi} B_\eta$ by the minimality of θ_0. Then if II plays this a_0 he beats I's strategy τ, a contradiction.

Lemma III. ($V = L[\mathcal{R}]$). Every hierarchy $\{\Gamma_i\}$ of Type III is of character Π and every Γ_i with $i \geqslant 2$ has the prewellordering property.

Proof. Γ_1 is such that Δ_1 is strongly closed. By 3.3 either Γ_1 or $\breve{\Gamma}_1$ has the reduction property. We claim that $\breve{\Gamma}_1$ cannot have the reduction property which will complete the proof that $\{\Gamma_i\}$ is of character $\boldsymbol{\Pi}$. In fact we show that Γ_1 cannot have the separation property. To see this let $P, Q \subseteq \mathcal{R}$ be two disjoint sets in $\exists^{\mathcal{R}} \Gamma_1 = \Gamma_2$. By repeating the argument in the preceding remark we conclude that P, Q can be separated by a set in Δ_1. Since $\Gamma_1 \cup \breve{\Gamma}_1 \subseteq \Gamma_2$ this clearly implies $\Gamma_1 \subseteq \Delta_1$, a contradiction.

To prove now that Γ_2 (and therefore all Γ_i for $i \geqslant 2$) has the prewellordering property we show that

$$\Gamma_2 = \{\bigcup_{\xi < \delta_1} B_\xi : B_\xi \in \Delta_1,\ \forall \xi < \delta_1\},$$

where $\delta_1 = \sup \{ \theta : \theta$ is the length of a prewellordering in $\Delta_1 \}$. (A similar idea was used in the proof of the main result in Kechris–Solovay [K–S]). The proof is analogous to arguments presented in the two previous lemmas and we leave the details to the reader.

Remark. Let us call (retaining the preceding notation) a sequence $\{B_\xi\}_{\xi < \delta_1}$ of sets in Δ_1 bounded if for every $B \in \Delta_1$

$$B \subseteq \bigcup_{\xi < \delta_1} B_\xi \Rightarrow \exists \theta < \delta_1 \, (B \subseteq \bigcup_{\xi < \theta} B_\xi).$$

It is easy to check that if Γ_1 has the prewellordering property then

$\Gamma_1 = \{ \bigcup_{\xi < \delta_1} B_\xi : B_\xi \in \Delta_1$ for all $\xi < \delta_1$ and $\{B_\xi\}_{\xi < \delta_1}$ is bounded$\}$.

Conversely, one can see that if Δ_1 is not closed under bounded unions i.e. if there is bounded $\{B_\xi\}_{\xi < \delta_1} \subseteq \Delta_1$ such that $\bigcup_{\xi < \delta_1} B_\xi \notin \Delta_1$, then

$\Gamma_1 = \{ \bigcup_{\xi < \delta_1} B_\xi : B_\xi \in \Delta_1, \forall \xi < \delta_1$ and $\{B_\xi\}_{\xi < \delta_1}$ is bounded$\}$ and this latter class can be easily checked to have always the prewellordering property, so that Γ_1 has the prewellordering property. Thus Γ_1 has the prewellordering property iff Δ_1 is not closed under bounded unions.

Lemma IV. ($V = L[\mathcal{R}]$). Every projective – like hierarchy of Type IV is of character Σ and every pointclass in the hierarchy has the prewellordering property.

Proof. Let Γ be as in the definition of Type IV hierarchy. Then by Kechris–Solovay [K–S], either Γ or $\breve{\Gamma}$ has the prewellordering property. Assume without loss of generality that Γ does. Consider $\delta = \sup \{ \xi : \xi$ is the length of a prewellordering in $\Delta \}$. Then there is a sequence $\{A_\xi\}_{\xi < \delta}$ of sets in Δ whose union is in $\Gamma - \Delta$ so if we let

$$\Gamma^* = \{ \bigcup_{\xi < \delta} B_\xi : B_\xi \in \breve{\Gamma}, \ \forall \xi < \delta \}$$

then $\Gamma^* \supseteq \Gamma$, $\Gamma^* \supseteq \breve{\Gamma}$ and since Γ^* is closed under $\exists^{\mathcal{R}}$ clearly $\Gamma^* \supseteq \breve{\Gamma}_1$. But by Moschovakis [Mo 1] $\Gamma^* \subseteq \breve{\Gamma}_1$ therefore $\Gamma^* = \Gamma_1$. But Γ^* has the prewellordering property which completes the proof. Indeed, if $A = \bigcup_{\xi < \delta} B_\xi$, where $B_\xi \in \breve{\Gamma}$ for each $\xi < \delta$ and we let as usual for $a \in A$

$$\sigma(a) = \text{least } \xi < \delta \text{ such that } a \in B_\xi,$$

then σ is a $\breve{\Gamma}_1$ – norm because for the associated relations we have

$$x \leqslant^*_\sigma \beta \Leftrightarrow \exists \xi < \delta [x \in B_\xi \wedge \forall \eta < \xi (B \notin B_\eta)]$$

$$x <^*_\sigma \beta \Leftrightarrow \exists \xi < \delta [x \in A_\xi \wedge \forall \eta \leqslant \xi (B \notin B_\eta)],$$

so that they both are in $\breve{\Gamma}_1$, since $\breve{\Gamma}_1$ is closed under unions of sequences of length δ by Moschovakis [Mo 1] and for each $\xi < \delta$, $\bigcap_{\eta < \xi} (\neg B_\eta) = \neg (\bigcup_{\eta < \xi} B_\eta) \in \Gamma$, since $\breve{\Gamma}$ is closed under $< \delta$ unions by Moschovakis [Mo 1] again.

REFERENCES

[A – M] J. W. Addison and Y.N. Moschovakis, Some consequences of the axiom of definable determinateness, Proc. Nat. Acad. Sci. U.S.A. 59 (1968), 708–712.

[B – M] J. Burgess and D. Miller, Remarks on Invariant Descriptive Set Theory, Fund. Math., XC (1975), 53–75.

[K – S] A.S. Kechris and R.M. Solovay, Projective algebras and the prewellordering property in L[R], mimeographed notes, 1976.

[Ma 1] D. A. Martin, The axiom of determinateness and reduction principles in the analytical hierarchy , Bull. Amer. Math. Soc. 74 (1968), 687–689.

[Ma 2] D.A. Martin, Projective sets and cardinal numbers; some questions related to the continuum problem, Jour. Symb. Logic, to appear.

[Ma 3] D. A. Martin, The Wadge ordering is a wellordering, mimeographed note.

[Mo 1] Y. N. Moschovakis, Determinacy and prewellorderings of the continuum, Math. Logic & Fund. of set theory (Proc. Intern. Colloq., Jerusalem 1968), North Holland, Amsterdam, 1970, 24–62.

[Mo 2] Y. N. Moschovakis, Elementary Induction on Abstract Structures, North Holland, Amsterdam, 1974.

[St] J. Steel, Ph. D. Thesis, Berkeley 1977.

[V 1] R. Van Wesep, Wadge degrees and descriptive set theory, Caltech–UCLA Logic Seminar 1977, Lectures Notes in Mathematics, Springer – Verlag, to appear.

[V 2] R. Van Wesep, Separation principles and the axion of determinateness, Jour. Sumb. Logic, 43 (1978), 77–81.

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